ON MODULAR INEQUALITIES IN VARIABLE L^p SPACES

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Abstract. We show that the Hardy-Littlewood maximal operator and a class of Calderón-Zygmund singular integrals satisfy the strong type modular inequality in variable L^p spaces if and only if the variable exponent $p(x) \sim const$.

1. Introduction

Let $p : \mathbb{R}^n \to [1, \infty)$ be a measurable function. Denote by $L^{p(\cdot)}(\mathbb{R}^n)$ the Banach space of measurable functions f on $\mathbb{R}^n$ such that for some $\lambda > 0$,

$$\int_{\mathbb{R}^n} |f(x)/\lambda|^{p(x)} dx < \infty,$$

with norm

$$\|f\|_{L^{p(\cdot)}} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} |f(x)/\lambda|^{p(x)} dx \leq 1 \right\}.$$

The spaces $L^{p(\cdot)}(\mathbb{R}^n)$ are a special case of Musielak-Orlicz spaces (cf. [9]). The behavior of some classical operators in harmonic analysis on $L^{p(\cdot)}(\mathbb{R}^n)$ is intensively investigated during several last years. Among numerous papers appeared in this area, let us mention only those of specific interest to us, to be precise those where different aspects concerning the boundedness on $L^{p(\cdot)}(\mathbb{R}^n)$ of the Hardy-Littlewood maximal operator [1, 2, 3, 10, 11, 12] and the Calderón-Zygmund operators [4, 8] were studied.

We recall that the Hardy-Littlewood maximal operator is defined for any $f \in L^1_{loc}(\mathbb{R}^n)$ by

$$Mf(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)| dy,$$

where the supremum is taken over all balls B containing x.

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Let $p_- = \operatorname{ess\,inf}_{x\in\mathbb{R}^n} p(x) > 1$ and $p_+ = \operatorname{ess\,sup}_{x\in\mathbb{R}^n} p(x) < \infty$. It has been proved by Diening [2] that if p satisfies the following uniform continuity condition

$$|p(x) - p(y)| \le \frac{c}{-\log|x-y|}, \quad |x-y| < 1/2, \tag{1.1}$$

and if p is a constant outside some large ball, then

$$\|Mf\|_{L^{p(\cdot)}} \le c\|f\|_{L^{p(\cdot)}} \tag{1.2}$$

for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$. After that the second condition on p has been improved independently in several directions by Cruz-Uribe, Fiorenza, and Neugebauer [1] and Nekvinda [10]. For example, it is shown in [1] that if p satisfies (1.1) and

$$|p(x) - p(y)| \le \frac{c}{\log(e+|x|)}, \quad |y| \ge |x|,$$

then (1.2) holds.

Diening and Růžička [4] (see also [8, Theorem 2.7] and [3, Section 8]) have proved that if $p_- > 1$ and $p_+ < \infty$, then a large class of Calderón-Zygmund operators is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ provided the Hardy-Littlewood maximal operator is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ and on $L^{(p(\cdot)/s)'}$ for some $0 < s < 1$, where $p'(x) = p(x)/(p(x)-1)$.

A natural question arises about conditions on p implying the strong type inequality

$$\int_{\mathbb{R}^n} |Rf(x)|^{p(x)} dx \le c \int_{\mathbb{R}^n} |f(x)|^{p(x)} dx \tag{1.3}$$

(so-called modular inequality in terminology of Musielak [9]), where R is any of the above-mentioned classical operators. Note that in [1] the weak type modular inequality

$$|\{x \in \mathbb{R}^n : Mf(x) > \alpha\}| \le c \int_{\mathbb{R}^n} |f(x)/\alpha|^{p(x)} dx \quad (\alpha > 0)$$

is proved under extremely weak assumptions on p. It is easy to see that (1.3) yields the norm inequality

$$\|Rf\|_{L^{p(\cdot)}} \le c\|f\|_{L^{p(\cdot)}},$$

and therefore one should expect that the class of functions p, for which (1.3) holds, must be smaller than the corresponding class implying (1.2). Nevertheless, our main result is somewhat surprising, since it says that this class is trivial. More precisely, we have the following.

Theorem 1.1. *Let $p_- > 1$ and $p_+ < \infty$. Then the inequality*

$$\int_{\mathbb{R}^n} (Mf(x))^{p(x)} dx \le c \int_{\mathbb{R}^n} |f(x)|^{p(x)} dx \tag{1.4}$$

holds for any $f \in L^{p(\cdot)}(\mathbb{R}^n)$ if and only if $p(x) \sim const.$

It is noteworthy that analogous questions on singular integrals are very similar to those when the boundedness on weighted L^p_ω implies $\omega \in A_p$ (cf. [13, p. 210]). We shall deal with a singular integral operator $Tf = f * K$, with kernel K satisfying the standard conditions

$$\|\widehat{K}\|_\infty \le c, \quad |K(x)| \le c/|x|^n,$$
$$|K(x) - K(x-y)| \le c|y|/|x|^{n+1} \text{ for } |y| < |x|/2,$$

and an additional nondegeneracy condition

$$|K(tu_0)| \ge c'/|tu_0|^n$$

for some unit vector u_0 and any $t \in \mathbb{R}$. Observe that this class of operators contains, for instance, any one of the Riesz transforms.

Theorem 1.2. *Let $p_- > 1$ and $p_+ < \infty$. Then the inequality*

$$\int_{\mathbb{R}^n} |Tf(x)|^{p(x)} dx \le c \int_{\mathbb{R}^n} |f(x)|^{p(x)} dx \tag{1.5}$$

holds for any $f \in L^{p(\cdot)}(\mathbb{R}^n)$ if and only if $p(x) \sim const.$

2. Proofs

By a weight we mean any non-negative locally integrable function on $\mathbb{R}^n$. Given a ball B and $f \in L^1_{loc}(\mathbb{R}^n)$, let $f_B = |B|^{-1} \int_B f$. For measurable f and g the notation $f \sim g$ means $f(x) = g(x)$ a.e.

We say that a weight ω satisfies A_∞ Muckenhoupt's condition if for any α, $0 < \alpha < 1$, there exists a β, $0 < \beta < 1$, such that $|E| \ge \alpha |B|$ implies $\int_E \omega dx \ge \beta \int_B \omega dx$ for all balls B and all subsets $E \subset B$. There are many equivalent characterizations of A_∞ (see, e.g., [13, Ch. 5]). In particular, $\omega \in A_\infty$ if and only if (see [6, p. 405] or [7])

$$\left(\frac{1}{|B|}\int_B \omega dx\right) \exp\left(\frac{1}{|B|}\int_B \log(1/\omega) dx\right) \le A. \tag{2.1}$$

We say that a family of weights $\{\omega_\alpha\}_{\alpha \in \mathcal{A}}$ satisfies A_∞ condition uniformly in α if $\omega_\alpha \in A_\infty$ for any $\alpha \in \mathcal{A}$ with corresponding A_∞ constants independing of α.

Lemma 2.1. *Let p be a non-negative measurable function on $\mathbb{R}^n$. The family of weights $\{t^{p(x)}\}_{t>0}$ satisfies A_∞ condition uniformly in t if and only if $p(x) \sim const.$*

Proof. When $p(x) \sim const$ the statement of the lemma is trivial. Thus, we assume that $t^{p(x)} \in A_\infty$ uniformly in t. Applying (2.1) to $\omega_t(x) = t^{p(x)}$ yields

$$\frac{1}{|B|}\int_B t^{p(x)-p_B}dx \le A \tag{2.2}$$

for any ball B and all $t > 0$. Now, if $|\{x \in B : p(x) > p_B\}| > 0$, we get a contradiction by letting $t \to \infty$ in (2.2). Analogously, if $|\{x \in B : p(x) < p_B\}| > 0$, we get a contradiction by letting $t \to 0$ in (2.2). Therefore, $p(x) = p_B$ for a.e. $x \in B$ and for all balls B. Hence, the limit $p_\infty = \lim_{|B|\to\infty} p_B$, where it is taken over all balls B in $\mathbb{R}^n$ as the measure $|B|$ tends to infinity, exists, and $p(x) = p_\infty$ for a.e. $x \in \mathbb{R}^n$. □

We are now in a position to prove Theorems 1.1 and 1.2. Since for $p(x) \sim const$ both theorems represent known classical results, we need to prove only the converse directions.

Proof of Theorem 1.1. It follows from (1.4) that for any ball B and any $f \in L^1_{loc}(\mathbb{R}^n)$,

$$\int_B (|f|_B)^{p(x)}dx \le c\int_B |f(x)|^{p(x)}dx. \tag{2.3}$$

Let $E \subset B$ be an arbitrary measurable subset with $|E| \ge \alpha|B|$, $0 < \alpha < 1$. Taking in (2.3) $f = t\chi_E$, $t > 0$, we get

$$\alpha^{p_+}\int_B t^{p(x)}dx \le c\int_E t^{p(x)}dx.$$

Therefore, the family of weights $\{t^{p(x)}\}_{t>0}$ satisfies A_∞ condition uniformly in t. Now we invoke Lemma 2.1 to complete the proof. □

Proof of Theorem 1.2. We use the following property of singular integrals (see [13, Ch. 5, 4.6]): for any ball B there exists a ball B' such that $f_B \le c|(Tf)\chi_{B'}|$ for any non-negative $f \in C_0^\infty$ supported in B and $f_{B'} \le c|(Tf)\chi_B|$ for any non-negative $f \in C_0^\infty$ supported in B'. It follows from this and from (1.5) that for such f,

$$\int_{B'} (f_B)^{p(x)}dx \le c'\int_B (f(x))^{p(x)}dx \tag{2.4}$$

and

$$\int_B (f_{B'})^{p(x)}dx \le c'\int_{B'} (f(x))^{p(x)}dx. \tag{2.5}$$

A simple limiting argument extends these estimates for any $f \geq 0$. Taking in (2.4) $f = t\chi_E$, $t > 0$, where $E \subset B$ with $|E| \geq \alpha|B|$, we get

$$\alpha^{p+} \int_{B'} t^{p(x)} dx \leq c' \int_E t^{p(x)} dx.$$

However (2.5), with $f = t\chi_{B'}$, yields

$$\int_B t^{p(x)} dx \leq c' \int_{B'} t^{p(x)} dx,$$

and therefore,

$$\alpha^{p+} \int_B t^{p(x)} dx \leq c'^2 \int_E t^{p(x)} dx.$$

This gives the desired result exactly as in the previous proof. □

3. Concluding remarks

Remark 3.1. We recall that a weight ω is doubling if there exists a constant $c > 0$ such that $\int_{2B} \omega dx \leq c \int_B \omega dx$ for any ball $B \subset \mathbb{R}^n$. It is well known that any A_∞ weight is doubling but the converse is not true. In Lemma 2.1, the A_∞ condition, in general, can not be replaced by a wider doubling condition. Indeed, one can construct on the real line disjoint sets E_1 and E_2 of positive measure whose union is $\mathbb{R}^1$, while χ_{E_1} and χ_{E_2} are doubling measures (see [13, Ch. 1, 8.8]). Let now $p(x) = c_1\chi_{E_1\times\mathbb{R}^{n-1}} + c_2\chi_{E_2\times\mathbb{R}^{n-1}}$, where $c_1 \neq c_2$. Then $p(x) \not\sim const$, while it is easy to check that $\int_{2B} t^{p(x)} dx \leq c \int_B t^{p(x)} dx$ for any ball $B \subset \mathbb{R}^n$ and all $t > 0$.

However, assuming additionally that p is continuous, one can show that the family $\{t^{p(x)}\}_{t>0}$ is doubling uniformly in t if and only if $p(x) = const$.

Remark 3.2. Musielak-Orlicz spaces (cf. [9]) consist of all measurable f such that for some $\lambda > 0$,

$$\int_{\mathbb{R}^n} \varphi(x, |f(x)/\lambda|) dx < \infty,$$

where $\varphi : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies specific conditions. These spaces contain, i.e., weighted Lebesgue spaces L^p_ω (when $\varphi(x,\xi) = \xi^p\omega(x)$) and Orlicz spaces (when $\varphi(x,\xi)$ is constant in the first variable).

Theorems 1.1 and 1.2 show that in the case $\varphi(x,\xi) = \xi^{p(x)}$ the corresponding modular inequality for M or T holds iff φ is constant in the first variable. It is easy to see that in general an analogous result does not hold. For example, one can take $\varphi(x,\xi) = \xi^p\omega(x)$ with ω satisfying the A_p Muckenhoupt condition.

On the other hand, it is known (see, e.g., [5]) that in the context of Orlicz spaces the modular inequality for M is equivalent to the norm inequality. Theorem 1.1 shows that this is not the case in the context of Musielak-Orlicz spaces.

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